The Baer Product and Extensions of Hopf Orders

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1. Introduction

Let p be prime and let K be a field of characteristic p that is complete with respect to a discrete valuation $\nu : K \to \mathbb{Z} \cup \{\infty\}$. Let R denote the valuation ring with unique maximal ideal $\mathfrak{m} = (\pi)$, $\nu(\pi) = 1$.

Let C_p^n denote the elementary abelian group of order p^n and let C_{p^n} denote the cyclic group of order p^n for $n = 1, 2, 3$.

For $i_1, i_2 \geq 0$ integers, $C_p = \langle g_1 \rangle$, let

$$E(i_1) = R\left[\frac{g_1 - 1}{\pi^{i_1}}\right] \text{ and } E(i_2) = R\left[\frac{g_1 - 1}{\pi^{i_2}}\right]$$

be Hopf orders in $K[C_p]$.

Let $\mathcal{E}(E(i_2), E(i_1))$ denote the set of equivalence classes of short exact sequences of Hopf orders

$$R \longrightarrow E(i_1) \xrightarrow{j} H \xrightarrow{s} E(i_2) \longrightarrow R. \tag{1}$$

We can endow $\mathcal{E}(E(i_2), E(i_1))$ with the Baer product $*$, so that $\mathcal{E}(E(i_2), E(i_1))$ is a group.

There is a subgroup $\mathcal{E}_{gt}(E(i_2), E(i_1))$ of $\mathcal{E}(E(i_2), E(i_1))$, consisting of the *generically trivial extensions*, i.e., those extensions of the form (1) which after tensoring with K, appear as

$$K \longrightarrow K[C_p] \xrightarrow{j} K[C_p \times C_p] \xrightarrow{s} K[C_p] \longrightarrow K. \qquad (2)$$

G. G. Elder and U (2017) have classified the subgroup $\mathcal{E}_{gt}(E(i_2), E(i_1))$.

Elements of $\mathcal{E}_{gt}(E(i_2), E(i_1))$ appear as

$$E_\mu : \; R \longrightarrow E(i_1) \xrightarrow{j} E(i_1, i_2, \mu) \xrightarrow{s} E(i_2) \longrightarrow R,$$

where the middle term is a *truncated exponential Hopf order* in $K[C_p \times C_p]$ of the form

$$E(i_1, i_2, \mu) = R\left[\frac{g_1 - 1}{\pi^{i_1}}, \frac{g_2 g_1^{[\mu]} - 1}{\pi^{i_2}}\right].$$

Here μ is an element of K that satisfies the valuation condition $\nu(\wp(\mu)) \geq i_2 - p i_1$, and $g_1^p = g_2^p = 1$.

Passing to the cyclic case, let D denote an arbitrary R-Hopf order in $K[C_{p^2}]$, $C_{p^2} = \langle g_2 \rangle$, $g_2^p = g_1$.

From the short exact sequence of groups

$$\{1\} \longrightarrow \langle g_2^p \rangle \xrightarrow{j} C_{p^2} \xrightarrow{s} \langle \overline{g}_2 \rangle \longrightarrow \{1\}, \qquad (3)$$

we obtain a short exact sequence of K-Hopf algebras,

$$K \longrightarrow K[C_p] \xrightarrow{j} K[C_{p^2}] \xrightarrow{s} K[C_p] \longrightarrow K. \qquad (4)$$

Since is an D is R-Hopf order in $K[C_{p^2}]$, from (4) we obtain a short exact sequence of R-Hopf orders

$$E: \; R \longrightarrow E(i_1) \xrightarrow{j} D \xrightarrow{s} E(i_2) \longrightarrow R,$$

where

$$E(i_1) = R\left[\frac{g_2^p - 1}{\pi^{i_1}}\right] \text{ and } E(i_2) = R\left[\frac{\overline{g}_2 - 1}{\pi^{i_2}}\right]$$

are R-Hopf orders in $K[C_p]$.

Because $\mathrm{char}(K) = p$, we must have $pi_2 \leq i_1$.

Consequently, there is a *distinguished extension*

$$E_0: \ R \longrightarrow E(i_1) \xrightarrow{j} R\left[\frac{g_1 - 1}{\pi^{i_1}}, \frac{g_2 - 1}{\pi^{i_2}}\right] \xrightarrow{s} E(i_2) \longrightarrow R$$

whose middle term is an R-Hopf order in $K[C_{p^2}]$ (a Larson order).

In the group $\langle \mathcal{E}(E(i_2), E(i_1)), * \rangle$, the inverse of E_0 is

$$E_0^{-1}: \ R \longrightarrow E(i_1) \xrightarrow{j} R\left[\frac{g_1^{p-1} - 1}{\pi^{i_1}}, \frac{g_2 - 1}{\pi^{i_2}}\right] \xrightarrow{s} E(i_2) \longrightarrow R,$$

with $g_2^p = g_1^{p-1}$.

Thus, under the Baer product,

$$[E] * [E_0^{-1}]$$

is a generically trivial extension in $\mathcal{E}_{gt}(E(i_2), E(i_1))$ and is thus of the form $[E_\mu]$ for some $\mu \in K$.

And so,

$$[E] = [E_\mu] * [E_0].$$

In this manner, we can classify E; the middle term of E is

$$D = R\left[\frac{g_1 - 1}{\pi^{i_1}}, \frac{g_2 g_1^{[\mu]} - 1}{\pi^{i_2}}\right], \; g_2^p = g_1, g_1^p = 1,$$

which is an R-Hopf order in $K[C_{p^2}]$.

So, in this way we obtain a complete classification of R-Hopf orders in $K[C_{p^2}]$.

(And thus, recover a result of D. Tossici (2010).)

2. The Baer Product, I

The following discussion of the Baer product was outlined in [Ch...21, Section 12.6.1].

Let H, H' be commutative, cocommutative R-Hopf algebras and let $\mathcal{E}(H', H)$ denote the set of equivalence classes of short exact sequences of R-Hopf algebras; $\mathcal{E}(H', H)$ contains the extensions of H by H'.

On $\mathcal{E}(H', H)$ we define a multiplication as follows. Let

$$E_1 : \; R \to H \stackrel{j_1}{\to} H_1 \stackrel{s_1}{\to} H' \to R,$$

$$E_2 : \; R \to H \stackrel{j_2}{\to} H_2 \stackrel{s_2}{\to} H' \to R,$$

be short exact sequences of R-Hopf algebras.

Since the tensor product of two Hopf algebras is again a Hopf algebra, we obtain a short exact sequence of R-Hopf algebras,

$$R \to H \otimes_R H \overset{j_1 \otimes j_2}{\to} H_1 \otimes_R H_2 \overset{s_1 \otimes s_2}{\to} H' \otimes_R H' \to R,$$

$(j_1 \otimes j_2)(a \otimes b) = j_1(a) \otimes j_2(b)$, $(s_1 \otimes s_2)(x \otimes y) = s_1(x) \otimes s_2(y)$,

Let the pair of morphisms $\alpha : A \to H_1 \otimes_R H_2$, $\beta : A \to H'$ be the pull-back of $(s_1 \otimes s_2, \Delta_{H'})$, that is,

$$A = \{(\sum_i x_i \otimes y_i) \otimes z \in H_1 \otimes H_2 \otimes H' \mid (s_1 \otimes s_2)(\sum_i x_i \otimes y_i) = \Delta_{H'}(z)\},$$

$\alpha((\sum_i x_i \otimes y_i) \otimes z) = \sum_i x_i \otimes y_i$ and $\beta((\sum_i x_i \otimes y_i) \otimes z) = z$.

Then there is a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
R & \to & H \otimes_R H & \stackrel{j_1 \otimes j_2}{\to} & H_1 \otimes_R H_2 & \stackrel{s_1 \otimes s_2}{\to} & H' \otimes_R H' & \to & R \\
\| & & \| & & \alpha \uparrow & & \Delta_{H'} \uparrow & & \| \\
R & \to & H \otimes_R H & \stackrel{j}{\to} & A & \stackrel{\beta}{\to} & H' & \to & R
\end{array}$$

In fact, A is an R-Hopf algebra. As evidence...

Proposition 1.

Let $m_{H_1 \otimes H_2 \otimes H'}$ denote multiplication in $H_1 \otimes H_2 \otimes H'$ and let $\Delta_{H_1 \otimes H_2 \otimes H'}$ denote comultiplication in $H_1 \otimes H_2 \otimes H'$. Then

(i) $m_{H_1 \otimes H_2 \otimes H'}(A \otimes A) \subseteq A$,

(ii) $\Delta_{H_1 \otimes H_2 \otimes H'}(A) \subseteq A \otimes A$.

Proof

For (i): Let $(\sum_k a_k \otimes b_k) \otimes c$, $(\sum_i x_i \otimes y_i) \otimes z$ be elements of A.

Then $(s_1 \otimes s_2)(\sum_k a_k \otimes b_k) = \Delta_{H'}(c)$ and $(s_1 \otimes s_2)(\sum_i x_i \otimes y_i) = \Delta_{H'}(z)$. Thus

$$(s_1 \otimes s_2)(\sum_k \sum_i a_k x_i \otimes b_k y_i) = \Delta_{H'}(cz).$$

For (ii): From $(s_1 \otimes s_2)(\sum_i x_i \otimes y_i) = \Delta_{H'}(z)$, we obtain

$$\Delta_{H' \otimes H'}(s_1 \otimes s_2)(\sum_i x_i \otimes y_i) = \Delta_{H' \otimes H'} \Delta_{H'}(z).$$

Now, the LHS is equal to

$$\begin{aligned}
&((s_1 \otimes s_2) \otimes (s_1 \otimes s_2))\Delta_{H_1 \otimes H_2}(\sum_i x_i \otimes y_i) \\
= \;& ((s_1 \otimes s_2) \otimes (s_1 \otimes s_2)) \sum_i \sum_{(x_i),(y_i)} x_{i(1)} \otimes y_{i(1)} \otimes x_{i(2)} \otimes y_{i(2)} \\
= \;& \sum_i \sum_{(x_i),(y_i)} (s_1 \otimes s_2)(x_{i(1)} \otimes y_{i(1)}) \otimes (s_1 \otimes s_2)(x_{i(2)} \otimes y_{i(2)}).
\end{aligned}$$

And the RHS is equal to

$$\begin{aligned}
&(I_{H'} \otimes \tau \otimes I_{H'})(\Delta_{H'} \otimes \Delta_{H'})\Delta_{H'}(z) \\
&= (I_{H'} \otimes \tau \otimes I_{H'}) \sum_{(z)} \Delta_{H'}(z_{(1)}) \otimes \Delta_{H'}(z_{(2)}) \\
&= (I_{H'} \otimes \tau \otimes I_{H'}) \sum_{(z),(z_{(1)}),(z_{(2)})} {z_{(1)}}_{(1)} \otimes {z_{(1)}}_{(2)} \otimes {z_{(2)}}_{(1)} \otimes {z_{(2)}}_{(2)} \\
&= \sum_{(z),(z_{(1)}),(z_{(2)})} {z_{(1)}}_{(1)} \otimes {z_{(2)}}_{(1)} \otimes {z_{(1)}}_{(2)} \otimes {z_{(2)}}_{(2)} \\
&= \sum_{(z),(z_{(1)}),(z_{(2)})} {z_{(1)}}_{(1)} \otimes {z_{(1)}}_{(2)} \otimes {z_{(2)}}_{(1)} \otimes {z_{(2)}}_{(2)}.
\end{aligned}$$

The last equality holds since H is cocommutative.

Thus

$$\sum_i \sum_{(x_i),(y_i)} (s_1 \otimes s_2)(x_{i(1)} \otimes y_{i(1)}) \otimes (s_1 \otimes s_2)(x_{i(2)} \otimes y_{i(2)})$$

$$= \sum_{(z),(z_{(1)}),(z_{(2)})} z_{(1)(1)} \otimes z_{(1)(2)} \otimes z_{(2)(1)} \otimes z_{(2)(2)}$$

$$= \sum_{(z)} \Delta_{H'}(z_{(1)}) \otimes \Delta_{H'}(z_{(2)}).$$

Hence,

$$(\sum_i x_{i(1)} \otimes y_{i(1)}) \otimes z_{(1)} \in A$$

and

$$(\sum_i x_{i(2)} \otimes y_{i(2)}) \otimes z_{(2)} \in A.$$

To finish the proof of (ii) , let $\Theta = (I_{H_1 \otimes H_2} \otimes \tau \otimes I_{H'})$. Then

$$\Delta_{H_1 \otimes H_2 \otimes H'}((\sum_i x_i \otimes y_i) \otimes z)$$

$$\begin{aligned}
&= \Theta(\Delta_{H_1 \otimes H_2} \otimes \Delta_{H'})((\sum_i x_i \otimes y_i) \otimes z) \\
&= \Theta \sum_i \sum_{(x_i),(y_i),(z)} x_{i(1)} \otimes y_{i(1)} \otimes (x_{i(2)} \otimes y_{i(2)}) \otimes z_{(1)} \otimes z_{(2)} \\
&= \sum_i \sum_{(x_i),(y_i),(z)} (x_{i(1)} \otimes y_{i(1)} \otimes z_{(1)}) \otimes (x_{i(2)} \otimes y_{i(2)} \otimes z_{(2)}) \\
&\in A \otimes A,
\end{aligned}$$

as required.

□

3. The Baer Product, II

Let

$$R \to H \otimes_R H \xrightarrow{j} A \xrightarrow{\beta} H' \to R$$

be the short exact sequence as constructed in Part I.

Let $m : H \otimes_R H \to H$ denote multiplication in H and let the pair of morphisms $\varrho : H \to B$, $i : A \to B$ be the push-out of (m, j), that is,

$$B = (H \otimes A)/S$$

with

$$S = \{m(x \otimes y) \otimes 1 - 1 \otimes j(x \otimes y) \in H \otimes A \mid x \otimes y \in H \otimes H\},$$

$\varrho(h) = (h \otimes 1) + S$ and $i(a) = (1 \otimes a) + S$.

There is a commutative diagram with exact rows:

$$\begin{array}{ccccccccccc} & R & \to & H \otimes_R H & \xrightarrow{j} & A & \xrightarrow{\beta} & H' & \to & R \\ & \| & & m \downarrow & & i \downarrow & & \| & & \| \\ E: & R & \to & H & \xrightarrow{\varrho} & B & \to & H' & \to & R \end{array}$$

The bottom row E is a short exact sequence of Hopf algebras.

Let $[E]$ be the equivalence class of E, which is an element of $\mathcal{E}(H', H)$. Let $[E_1]$, $[E_2]$ be the classes of E_1, E_2, respectively.

Then $[E]$ is the *Baer product* $*$ of classes of extensions;

$$[E] = [E_1] * [E_2].$$

We know that $H \otimes A$ is an R-Hopf algebra. In order for the quotient

$$B = (H \otimes A)/S$$

to be a Hopf algebra, S should be a Hopf ideal, that is, S is a biideal (ideal + coideal) that satisfies $\sigma_{H\otimes A}(S) \subseteq S$.

We prove the coideal property under the very special conditions that $H = E(i_1)$, $H' = E(i_2)$ are R-Hopf orders in $K[C_p]$, and $H_1 = E(i_1, i_2, \mu)$ and $H_2 = E(i_1, i_2, \gamma)$ are R-Hopf orders in $K[C_{p^2}]$, with $\langle g_2 \rangle = C_{p^2}$, $g_2^p = g_1$.

In this case,

$$A \subseteq E(i_1, i_2, \mu) \otimes E(i_1, i_2, \gamma) \otimes E(i_2),$$

$$H \otimes A = E(i_1) \otimes A.$$

Proposition 2.

S is a coideal of $E(i_1) \otimes A$, that is, $\varepsilon_{E(i_1)\otimes A}(S) = 0$ and

$$\Delta_{E(i_1)\otimes A}(S) \subseteq S \otimes (E(i_1) \otimes A) + (E(i_1) \otimes A) \otimes S,$$

Proof. Let

$$h = g_1^2 \otimes 1 \otimes 1 \otimes 1 - 1 \otimes g_1 \otimes g_1 \otimes 1.$$

Then h is an element of $S \subseteq E(i_1) \otimes A$. We have $\varepsilon_{E(i_1)\otimes A}(h) = 0$. So it remains to show that

$$\Delta_{E(i_1)\otimes A}(h) \in S \otimes (E(i_1) \otimes A) + (E(i_1) \otimes A) \otimes S.$$

We have

$$\Delta_{E(i_1)\otimes A}(g_1^2 \otimes 1 \otimes 1 \otimes 1)$$

$$\begin{aligned} &= (I_{E(i_1)} \otimes \tau \otimes I_A)(\Delta_{E(i_1)} \otimes \Delta_A)(g_1^2 \otimes 1 \otimes 1 \otimes 1) \\ &= (I_{E(i_1)} \otimes \tau \otimes I_A)(g_1^2 \otimes g_1^2 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1) \\ &= g_1^2 \otimes 1 \otimes 1 \otimes 1 \otimes g_1^2 \otimes 1 \otimes 1 \otimes 1. \end{aligned}$$

On the other hand,

$$\Delta_{E(i_1)\otimes A}(1 \otimes g_1 \otimes g_1 \otimes 1)$$

$$\begin{aligned} &= (I_{E(i_1)} \otimes \tau \otimes I_A)(\Delta_{E(i_1)} \otimes \Delta_A)(1 \otimes g_1 \otimes g_1 \otimes 1) \\ &= (I_{E(i_1)} \otimes \tau \otimes I_A)(1 \otimes 1 \otimes \Delta_A(g_1 \otimes g_1 \otimes 1)) \\ &= (I_{E(i_1)} \otimes \tau \otimes I_A)(1 \otimes 1 \otimes g_1 \otimes g_1 \otimes 1 \otimes g_1 \otimes g_1 \otimes 1) \\ &= 1 \otimes g_1 \otimes g_1 \otimes 1 \otimes 1 \otimes g_1 \otimes g_1 \otimes 1. \end{aligned}$$

Thus

$$\begin{aligned} \Delta_{E(i_1)\otimes A}(h) = g_1^2 \otimes 1 \otimes 1 \otimes 1 \otimes g_1^2 \otimes 1 \otimes 1 \otimes 1 \\ - 1 \otimes g_1 \otimes g_1 \otimes 1 \otimes 1 \otimes g_1 \otimes g_1 \otimes 1. \end{aligned}$$

Now,

$$(g_1^2\otimes 1\otimes 1\otimes 1)\otimes(g_1^2\otimes 1\otimes 1\otimes 1)-(1\otimes g_1\otimes g_1\otimes 1)\otimes(1\otimes g_1\otimes g_1\otimes 1)$$

$$= \; g_1^2\otimes 1\otimes 1\otimes 1\otimes g_1^2\otimes 1\otimes 1\otimes 1-1\otimes g_1\otimes g_1\otimes 1\otimes g_1^2\otimes 1\otimes 1\otimes 1$$

$$+\,1\otimes g_1\otimes g_1\otimes 1\otimes g_1^2\otimes 1\otimes 1\otimes 1-1\otimes g_1\otimes g_1\otimes 1\otimes 1\otimes g_1\otimes g_1\otimes 1$$

$$= \; (g_1^2\otimes 1\otimes 1\otimes 1-1\otimes g_1\otimes g_1\otimes 1)\otimes g_1^2\otimes 1\otimes 1\otimes 1$$

$$+\,1\otimes g_1\otimes g_1\otimes 1\otimes(g_1^2\otimes 1\otimes 1\otimes 1-1\otimes g_1\otimes g_1\otimes 1),$$

which is in

$$S\otimes(E(i_1)\otimes A)+(E(i_1)\otimes A)\otimes S.$$

□

4. Application to Hopf orders: $\mathcal{E}_{gt}(E(i_2), E(i_1))$

As shown in Elder and U (2017), all of the elements in $\mathcal{E}_{gt}(E(i_2), E(i_1))$ have been classified.

For $x \in K$, let $\wp(x) = x^p - x$.

Proposition 3 (Elder, U).
The subgroup $\mathcal{E}_{gt}(E(i_2), E(i_1))$ is isomorphic to the additive subgroup of $K/(\mathbb{F}_p + \mathfrak{m}^{i_2 - i_1})$ represented by those elements $\mu \in K$ satisfying $\nu(\wp(\mu)) \geq i_2 - pi_1$.

In more detail: an element in $\mathcal{E}_{gt}(E(i_2), E(i_1))$ can be written as

$$E_\mu : \ R \longrightarrow E(i_1) \xrightarrow{j} R\left[\frac{g_1 - 1}{\pi^{i_1}}, \frac{g_2 g_1^{[\mu]} - 1}{\pi^{i_2}}\right] \xrightarrow{s} E(i_2) \longrightarrow R,$$

for some $\mu \in K$ with $\nu(\wp(\mu) \geq i_2 - pi_1$. Note: $g_1^p = g_2^p = 1$.

So we let E_μ, E_γ be two elements of $\mathcal{E}_{gt}(E(i_2), E(i_1))$ and compute the Baer product $[E_\mu] * [E_\gamma]$.

In this case, $H = E(i_1)$,

$$H_1 = R\left[\frac{g_1 - 1}{\pi^{i_1}}, \frac{g_2 g_1^{[\mu]} - 1}{\pi^{i_2}}\right],$$

$$H_2 = R\left[\frac{g_1 - 1}{\pi^{i_1}}, \frac{g_2 g_1^{[\gamma]} - 1}{\pi^{i_2}}\right],$$

$H' = E(i_2)$, and

$$A \subseteq R\left[\frac{g_1 - 1}{\pi^{i_1}}, \frac{g_2 g_1^{[\mu]} - 1}{\pi^{i_2}}\right] \otimes R\left[\frac{g_1 - 1}{\pi^{i_1}}, \frac{g_2 g_1^{[\gamma]} - 1}{\pi^{i_2}}\right] \otimes E(i_2).$$

Now as $g_1^{[\mu]}, g_1^{[\gamma]} \in E(i_1)$ and $s_1(g_1^{[\mu]}) = s_2(g_1^{[\gamma]}) = 1$, we have

$$g_1^{[\mu]} \otimes g_1^{[\gamma]} \otimes 1 \in A.$$

So in the quotient

$$B = (E(i_1) \otimes A)/S,$$

the quantity

$$m_{E(i_1)}(g_1^{[\mu]} \otimes g_1^{[\gamma]}) \otimes 1 \otimes 1 \otimes 1 = g_1^{[\mu+\gamma]} \otimes 1 \otimes 1 \otimes 1$$

is identified with the tensor

$$1 \otimes g_1^{[\mu]} \otimes g_1^{[\gamma]} \otimes 1 \in E(i_1) \otimes A.$$

Thus the Baer product $[E_\mu] * [E_\gamma]$ is

$$E_{\mu+\gamma}: \ R \longrightarrow E(i_1) \xrightarrow{j} R\left[\frac{g_1 - 1}{\pi^{i_1}}, \frac{g_2 g_1^{[\mu+\gamma]} - 1}{\pi^{i_2}}\right] \xrightarrow{s} E(i_2) \longrightarrow R,$$

which is an element of $\mathcal{E}_{gt}(E(i_2), E(i_1))$.

5. Application to Hopf orders: a Result of Tossici

Next, let $C_{p^2} = \langle g_1, g_2 \rangle$ with $g_2^p = g_1$. Let D be an arbitrary R-Hopf order in $K[C_{p^2}]$.

Then there is a short exact sequence

$$E: \; R \longrightarrow E(i_1) \xrightarrow{j} D \xrightarrow{s} E(i_2) \longrightarrow R,$$

where

$$E(i_1) = R\left[\frac{g_2^p - 1}{\pi^{i_1}}\right] \text{ and } E(i_2) = R\left[\frac{\overline{g}_2 - 1}{\pi^{i_2}}\right]$$

are R-Hopf orders in $K[C_p]$.

Proposition 4.

$pi_2 \leq i_1$.

Proof.

Let $E(i_1)^+$ denote the augmentation ideal of $E(i_1)$. Since

$$D/j(E(i_1)^+)D = E(i_2),$$

the lift of the generator $\frac{\overline{g}_2 - 1}{\pi^{i_2}} \in E(i_2)$ must appear as

$$\frac{g_2 - 1}{\pi^{i_2}} + h,$$

for some $h \in j(E(i_1)^+)D$.

As $\mathrm{char}(K) = p$, we obtain

$$\left(\frac{g_2 - 1}{\pi^{i_2}} + h\right)^p = \frac{g_1 - 1}{\pi^{p i_2}} \in E(i_1),$$

thus $p i_2 \leq i_1$.

□

Since $p i_2 \leq i_1$ (Proposition 4), there exists a distinguished extension

$$E_0 : \; R \longrightarrow E(i_1) \xrightarrow{j} R\left[\frac{g_1 - 1}{\pi^{i_1}}, \frac{g_2 - 1}{\pi^{i_2}}\right] \xrightarrow{s} E(i_2) \longrightarrow R,$$

$g_2^p = g_1$.

Proposition 5.

In the group $\langle \mathcal{E}(E(i_2), E(i_1)), * \rangle$*, the inverse of* E_0 *is*

$$E_0^{-1}: \; R \longrightarrow E(i_1) \xrightarrow{j} R\left[\frac{g_1^{p-1}-1}{\pi^{i_1}}, \frac{g_2-1}{\pi^{i_2}}\right] \xrightarrow{s} E(i_2) \longrightarrow R,$$

with $g_2^p = g_1^{p-1}$*.*

Proof.

We compute the Baer product $[E_0] * [E_0^{-1}]$. In this case,

$$g_2 \otimes g_2 \otimes \overline{g}_2 \in A.$$

And so,

$$(g_2 \otimes g_2 \otimes \overline{g}_2)^p = g_1 \otimes g_1^{p-1} \otimes 1 \in A.$$

Now in the quotient $B = (E(i_1) \otimes A)/S$, we have

$$\begin{aligned}(1 \otimes g_2 \otimes g_2 \otimes \overline{g}_2)^p &= 1 \otimes g_1 \otimes g_1^{p-1} \otimes 1 \\ &= g_1 g_1^{p-1} \otimes 1 \otimes 1 \otimes 1 \\ &= 1 \otimes 1 \otimes 1 \otimes 1,\end{aligned}$$

and the Baer product $[E_0] * [E_0^{-1}]$ is the trivial element

$$R \longrightarrow E(i_1) \xrightarrow{j} R\left[\frac{g_1 - 1}{\pi^{i_1}}, \frac{g_2 - 1}{\pi^{i_2}}\right] \xrightarrow{s} E(i_2) \longrightarrow R,$$

with $g_2^p = g_1^p = 1$.

$\square$

Proposition 6.

The Baer product $[E] * [E_0^{-1}]$ *is a generically trivial extension, that is,* $[E] * [E_0^{-1}] \in \mathcal{E}_{gt}(E(i_2), E(i_1))$, *thus*

$$[E] * [E_0^{-1}] = [E_\mu],$$

for some $\mu \in K$.

Proof.

Use the formula

$$K \otimes ([E_\mu] * [E_\gamma]) \cong [K \otimes E_\mu] * [K \otimes E_\gamma].$$

□

Proposition 7.

The extension E appears as

$$R \longrightarrow E(i_1) \xrightarrow{j} R\left[\frac{g_1 - 1}{\pi^{i_1}}, \frac{g_2 g_1^{[\mu]} - 1}{\pi^{i_2}}\right] \xrightarrow{s} E(i_2) \longrightarrow R,$$

for some $\mu \in K$ with $\nu(\wp(\mu) \geq i_2 - pi_1$, $g_2^p = g_1$, $g_1^p = 1$.

Proof.

Assuming Proposition 6, we have

$$([E] * [E_0^{-1}]) * [E_0] = [E_\mu] * [E_0],$$

for some $\mu \in K$. Thus

$$[E] = [E_\mu] * [E_0].$$

And the Baer product $[E_\mu] * [E_0]$ can be computed as

$$R \longrightarrow E(i_1) \xrightarrow{j} R\left[\frac{g_1 - 1}{\pi^{i_1}}, \frac{g_2 g_1^{[\mu]} - 1}{\pi^{i_2}}\right] \xrightarrow{s} E(i_2) \longrightarrow R,$$

$g_2^p = g_1$, $g_1^p = 1$, which is the extension E. □

6. Hopf orders in $K[C_{p^2} \times C_p]$, $K[C_p \times C_{p^2}]$, $K[C_{p^3}]$

Let $E(i_1, i_2, \mu)$ be an R-Hopf order in $K[C_p^2]$ and let $E(i_3)$ be an R-Hopf order in $K[C_p]$.

U (2022) has classified the generically trivial extensions $\mathcal{E}_{gt}(E(i_3), E(i_1, i_2, \mu))$.

Proposition 8.

The group $\mathcal{E}_{gt}(E(i_3), E(i_1, i_2, \mu))$ is isomorphic to the additive subgroup of

$$K^2/(\mathbb{F}_p(\mu, -1) + (\mathbb{F}_p + \mathfrak{m}^{i_3 - i_1}) \times \mathfrak{m}^{i_3 - i_2})$$

represented by pairs $(\alpha, \beta) \in K^2$ which satisfy $\nu(\wp(\alpha) + \wp(\mu)\beta) \geq i_3 - pi_1$ and $\nu(\wp(\beta)) \geq i_3 - pi_2$.

An element in $\mathcal{E}_{gt}(E(i_3), E(i_1, i_2, \mu))$ appears as

$$E_{\alpha,\beta}: \ R \to E(i_1, i_2, \mu) \to R\left[\frac{g_1 - 1}{\pi^{i_1}}, \frac{g_2 g_1^{[\mu]} - 1}{\pi^{i_2}}, \frac{g_3 g_1^{[\alpha]}(g_2 g_1^{[\mu]})^{[\beta]} - 1}{\pi^{i_3}}\right]$$

$$\to E(i_3) \to R.$$

The middle term is an R-Hopf order in $K[C_p^3]$. Here, $C_p^3 = \langle g_1, g_2, g_3 \rangle$, $g_1^p = g_2^p = g_3^p = 1$.

Our plan is to use the Baer product to compute extensions whose middle terms are Hopf orders in $K[C_{p^2} \times C_p]$, $K[C_p \times C_{p^2}]$, or $K[C_{p^3}]$.

For instance, if $g_1^p = g_2^p = 1, g_3^p = g_2$, then $\langle g_1, g_2, g_3 \rangle = C_p \times C_{p^2}$.

And if

$$\frac{g_2 - 1}{\pi^{p i_3}} \in E(i_1, i_2, \mu),$$

then there exists a distinguished extension

$$E_0 : \ R \to E(i_1, i_2, \mu) \to R\left[\frac{g_1 - 1}{\pi^{i_1}}, \frac{g_2 g_1^{[\mu]} - 1}{\pi^{i_2}}, \frac{g_3 - 1}{\pi^{i_3}}\right]$$
$$\to E(i_3) \to R.$$

Consequently, the Baer product $[E_{\alpha,\beta}] * [E_0]$ is an element of $\mathcal{E}(E(i_3), E(i_1, i_2, \mu))$, and its middle term is an R-Hopf order in $K[C_p \times C_{p^2}]$.

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